POISSON'S EQUATION ON A COMPACT (RIEMANN) SURFACE

KS

The following is exactly the discussion from [Donaldson - Riemann Surfaces], with some review and details.

1. Preface (Geometry)

The setting for us is a compact Riemann surface, which is a complex 1-dimensional manifold X (so 2 real dimensions). Picture a sphere ($S^2 = \mathbf{C}P^1 = \mathbf{C} \cup \{\infty\}$), a torus ($T^2 = S^1 \times S^1$), or more generally any genus g surface ($T^2 \# \ldots \# T^2$), perhaps sitting (embedded) in $\mathbf{R}^3$. (The classification of closed surfaces tells us these are the only possibilities, since complex manifolds are orientable.) The main dicussion that follows should only touch the complex structure of X, because our goal is to solve Poisson's equation

$$\Delta f = \rho \tag{1}$$

and the Laplace operator Δ is well known before any complex geometry. It is most classically defined for functions $f : \mathbf{R}^n \to \mathbf{R}$:

$$\Delta f := \text{div } \nabla f = \sum_{i=1}^{n} \frac{\partial^2}{\partial {x_i}^2} f.$$

For our situation of a surface we make two notes.

First, if we start with X a smooth manifold, then we cannot even write down these formulas for the Laplacian Δ. If f is a function on X, and we use local coordinates to write $\sum \partial_i^2 f$, then this quantity will depend on the choice of coordinates.

Similarly, the divergence and gradient operations are also undefined, (mainly the divergence is the problem). One option is to put a Riemannian metric on X, again this is another choice. Instead we will see that the complex structure of X gives us a laplacian without the need of a metric. The cost will be to dip into differential forms; as a 2nd order differential operator, the laplacian will increase the grading by 2

$$\Delta : \Omega^0_X \to \Omega^2_X.$$

So in equation (1) we will require that ρ be a 2-form.

Differential k-forms are objects which can be (and are meant to be) integrated over k-dimensional regions. So for our surface, we want to be able to integrate 2-forms over X. Since X is a complex manifold, it is orientable, so we are allowed to write down the integral $\int_X \rho$ for a 2-form ρ, and compactness of X ensures a finite value.

The second point about the laplacian is that we will use the other sign convention. That is, our laplacian will be modelled on $\Delta = -\sum \partial_i^2 f$. In terms of analysis, this makes the (linear) operator Δ, defined on the space of functions, a positive operator — the eigenvalues will all be positive.

The result we want to show is

Theorem 1.1. *Let X be a compact Riemann surface, and let ρ be a (real) 2-form on X. Then $\Delta f = \rho$ has a solution $f : X \to \mathbf{R}$ if and only if $\int_X \rho = 0$, and this solution is unique up to an additive constant.*

2. Defining the Laplacian (Linear Algebra)

2.1. Differential forms.

We will assume a little bit, but also review a little bit, about differential forms. Let us first view X as a real 2-dimensional surface.

The space of 0-forms Ω^0_X is just defined as $C^\infty(X)$, the smooth (real-valued) functions on X.

The space of 1-forms Ω^1_X consists of objects α, which in local coordinates x^i look like vector valued functions. We write

$$\alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \sum_{i=1}^n \alpha_i \mathrm{d}x^i.$$

Here the $\alpha_i = \alpha_i(p)$ are smooth functions in $p \in X$. For each p, they are the coefficients of the covector $\alpha(p)$ in the cotangent space T^*_pX, a vector space spanned by $\{\mathrm{d}x^i\}_{i=1\cdots n}$. That is, the $\mathrm{d}x^i$ above are the "basis vectors". Since for us X is 2-dimensional, each cotangent space T^*_pX is 2-dimensional, spanned by $\{\mathrm{d}x, \mathrm{d}y\}$.

The space of 2-forms Ω^2_X also consists of vector-valued-function-like objects ρ which, in local coordinates can be written at a point $p \in X$ as

$$\rho = \sum_{i,j} \rho_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j.$$

where the $\rho_{ij} = \rho_{ij}(p)$ are smooth functions in p. For each fixed p, they are the components of the 2-covector $\rho(p) \in \bigwedge^2 T^*_pX$, a vector space spanned by $\{\mathrm{d}x^i \wedge \mathrm{d}x^j\}_{i<j}$. We have $\mathrm{d}x^i \wedge \mathrm{d}x^j = -\mathrm{d}x^j \wedge \mathrm{d}x^i$. In fact, since X is 2-dimensional this vector space is 1-dimensional, spanned by $\mathrm{d}x \wedge \mathrm{d}y$.

We have the exterior derivative

$$\mathrm{d} : \Omega^k \to \Omega^{k+1}$$
$$\mathrm{d}\left(\sum_I a_I \mathrm{d}x^I\right) = \sum_{I,i} \frac{\partial a_I}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^I.$$

where I denotes a multi-index. For us we only need

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y \quad \text{locally for } f \in \Omega^0_X$$
$$\mathrm{d}(\alpha_x \mathrm{d}x + \alpha_y \mathrm{d}y) = \left(\frac{\partial \alpha_y}{\partial x} - \frac{\partial \alpha_x}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y \quad \text{locally for } \alpha_x \mathrm{d}x + \alpha_y \mathrm{d}y \in \Omega^1_X$$

2.2. Complex (co)tangent space.

For this section, the generalization is easy, so we will work on a complex n-manifold X.

Definition 2.1. Given a real vector space V, an (almost) complex structure on V is a linear map J such that

$$J : V \to V$$
$$J^2 = -\mathrm{Id}_V.$$

Lemma 2.2. *Such a map can only exist on an even dimensional space.*

This lemma is not necessary for later, but I am including it with the justification that we are "getting to know complex structures better".

(Maybe the least elegant) Proof. Let e_1 be some nonzero vector in V. We claim that $\{e_1, Je_1\}$ is linearly independent. Suppose $Je_1 = ae_1$ for some nonzero $a \in \mathbf{R}$. Then apply J to get $-e_1 = aJe_1 = a^2e_1$ or $a^2 = -1$, which cannot happen.

Suppose we have $B = \{e_1, Je_1, \ldots, e_{n-1}, Je_{n-1}\}$ is a linearly independent set. If this set spans V, then we are done. If not, then take another vector e_n not in the span of B, and we claim that we can again throw in Je_n. Suppose

$$a_ne_n + b_nJe_n = \sum_{i=1}^{n-1} a_ie_i + b_iJe_i$$

with a_i, b_i not all 0. Note that $b_n \neq 0$ because e_n is not in the span of B. Now write the left side as $(a_n + b_nJ)e_n$ and multiply the equation by $(a_n - b_nJ)$ to get

$$(a_n^2 + b_n^2)e_n = (a_n - b_nJ)\sum_{i=1}^{n-1} a_ie_i + b_iJe_i.$$

Since $a_n^2 + b_n^2 \neq 0$, this contradicts that e_n is not in the span of B again. □

This proves further that we can always choose a nice basis $\{e_1, Je_1, \ldots, e_n, Je_n\}$ for V. We will want to have a complex structure on T_pX. In any basis we could (in fact we kind of do) write down the matrix

$$J = \begin{pmatrix} 0 & -1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & -1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & -1 \\ & & & & & 1 & 0 \end{pmatrix} \tag{2}$$

but we want to make sure that we are pulling from the complex nature of X. We know holomorphic functions should have complex-linear derivatives so let's allow this to define a complex structure.

Derivatives of functions live in the cotangent space. For complex-valued functions (not necessarily holomorphic), consider the complexified cotangent space $T_p^*X^{\mathbf{C}} := T_p^*X \otimes \mathbf{C} = \mathrm{Hom}_{\mathbf{R}}(T_pX, \mathbf{C})$ (this just means we can have complex coefficients in front of the $\mathrm{d}x$ and $\mathrm{d}y$). For any complex structure J on T_pX, a real-linear map $A : T_pX \to \mathbf{C}$ is called complex-linear if $A(Jv) = iA(v)$.

Lemma 2.3. *There is a unique complex structure J on T_pX such that the derivative of any holomorphic function is complex-linear.*

Recall that holomorphic functions are defined on complex manifolds in the same way that smooth functions are defined for real smooth manifolds, by requiring that it is holomorphic in any chart.

Proof. We have $p \in X$. Choose local coordinates $z^j = x^j + iy^j = (x^j, y^j)$ using the chart in $\mathbf{C}$. These define a basis $\{\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^j}\}$ for T_pX. Define J to be

$$\begin{aligned} J\left(\frac{\partial}{\partial x^j}\right) &= \frac{\partial}{\partial y^j} \\ J\left(\frac{\partial}{\partial y^j}\right) &= -\frac{\partial}{\partial x^j}, \end{aligned}$$

i.e. the matrix given above.
A complex-valued function f, locally given by $u(x,y) + iv(x,y)$, is traditionally holomorphic if

$$\begin{aligned} u_{x^j} &= v_{y^j} \\ u_{y^j} &= -v_{x^j}. \end{aligned}$$

We need $\mathrm{d}f(Jv) = i\mathrm{d}f(v)$.

$$\begin{aligned} \mathrm{d}f(J\partial_{x^j}) &= \mathrm{d}u(\partial_{y^j}) + i\mathrm{d}v(\partial_{y^j}) = u_{y^j} + iv_{y^j} \\ i\mathrm{d}f(\partial_{x^j}) &= -\mathrm{d}v(\partial_{x^j}) + i\mathrm{d}u(\partial_{x^j}) = -v_{x^j} + iu_{x^j} \end{aligned}$$

If f is holomorphic, these two lines are equal for each j. Applying to ∂_{y^j} gives the same conditions. Conversely, if we are defining J and require that these two lines be equal for holomorphic functions, then we must have $J\partial_{x^j} = \partial_{y^j}$ for each j. Then

$$J\partial_{y^j} = J^2\partial_{x^j} = -\partial_{x^j}$$

fully defines J as a complex structure, hence the uniqueness. □

The uniqueness proves that the form of J (equation (2)) remains the same in different complex charts, which is maybe surprising. Looking at the proof, it seems to be for the same reason that the Cauchy-Riemann equations maintain their form under holomorphic change of coordinates.

The above lemma starts the insight into the splitting of $T_p^*X^{\mathbf{C}}$. Continuing,

Lemma 2.4. *If V has a complex structure, then every $\mathbf{R}$-linear map $A : (V, J) \to \mathbf{C}$ decomposes uniquely into a sum of a complex-linear part and a complex-anti-linear part. That is*

$$\begin{aligned} \mathrm{Hom}_{\mathbf{R}}(V, \mathbf{C}) &= (V^*)^{1,0} \oplus (V^*)^{0,1} \\ \textit{where} \quad V^{1,0} &:= \{A(Jv) = iA(v)\} \\ V^{0,1} &:= \{A(Jv) = -iA(v)\} \end{aligned}$$

and $(V^*)^{1,0} \cong \overline{(V^*)^{0,1}}$ as complex vector spaces via the map $v \mapsto \bar{v}$.

The "complex isomorphism" above means that the map $(V^*)^{1,0} \to (V^*)^{0,1}$ is a complex anti-isomorphism.

Proof. Let

$$\begin{aligned} A &= A^{1,0} + A^{0,1} \\ A^{1,0} &:= \frac{1}{2}(A - iAJ) \\ A^{0,1} &:= \frac{1}{2}(A + iAJ). \end{aligned}$$

Then

$$\begin{aligned} A^{1,0}(Jv) &= \frac{1}{2}(A(Jv) - iA(J^2v)) = \frac{1}{2}(i(-i)A(Jv) + iA(v)) = iA^{1,0}(v) \\ A^{0,1}(Jv) &= \frac{1}{2}(A(Jv) + iA(J^2v)) = \frac{1}{2}(-i(i)A(Jv) - iA(v)) = -iA^{0,1}(v). \end{aligned}$$

So $A \overset{\pi^{1,0}}{\longmapsto} A^{1,0}$ and $A \overset{\pi^{0,1}}{\longmapsto} A^{0,1}$ are the projections. If $A \in (V^*)^{1,0}$, then $A = A^{1,0}$. The conjugated linear map $\overline{A}$ is complex anti-linear:

$$\overline{A}(v) = \frac{1}{2}\left(\overline{A(v) - iA(Jv)}\right) = \frac{1}{2}\left(\overline{A}(v) + i\overline{A}(Jv)\right) = \overline{A}^{0,1}(v)$$

We applied it to a vector $v \in V$ to make it clear what happens to the J under conjugation (nothing). If we conjugate again, the same calculation sends us back to $A = A^{1,0}$. Conjugation is of course complex anti-linear. $\square$

So we've split the cotangent space at each point $p \in X$,

$$T_p^* X^{\mathbf{C}} = T_p^* X^{1,0} \oplus T_p^* X^{0,1}.$$

This splits the cotangent bundle, and hence the forms split

$$\Omega_X^1(\mathbf{C}) = \Omega_X^{1,0} \oplus \Omega_X^{0,1}$$

(where $\Omega_X(\mathbf{C})$ is the space of complex-valued differential forms). And thus lemma 2.3 says that if f is a holomorphic function, then $\mathrm{d}f(p)$ lies entirely inside $T_p^* X^{1,0}$. For this reason:

Definition 2.5. We call $T_p^* X^{1,0}$ and $T_p^* X^{0,1}$ the holomorphic and anti-holomorphic cotangent spaces respectively.

It would help to have a basis for each space. First, some dimension counting. If X is an n-dimensional complex manifold, then $T_p^* X$ is real $2n$-dimensional, $T_p^* X^{\mathbf{C}}$ is complex $2n$-dimensional, and so $T_p^* X^{1,0}$ and $T_p^* X^{0,1}$ are each complex n-dimensional. Note the coordinate function z is holomorphic, and the function $\bar{z}$ is anti-holomorphic, so

$$\begin{aligned} \mathrm{d}z^j &= \mathrm{d}x^j + i\mathrm{d}y^j \in T_p^* X^{1,0} \\ \mathrm{d}\bar{z}^j &= \mathrm{d}x^j - i\mathrm{d}y^j \in T_p^* X^{0,1} \end{aligned}$$

Thus we can write $T_p^*X^{1,0} = \mathrm{span}\{\mathrm{d}z^1, \ldots, \mathrm{d}z^n\}$, and similarly for $T_p^*X^{0,1}$. This also defines a dual basis $\left\{\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^j}\right\}$ for $T_pX^{\mathbf{C}}$, by the condition that $\mathrm{d}z^j(\partial_{z^k}) = \delta_k^j$, etc. We get

$$
\begin{aligned}
\mathrm{d}x^j(\partial_{z^k}) &= \frac{1}{2}(\mathrm{d}z^j + \mathrm{d}\bar{z}^j)(\partial_{z^k}) = \frac{1}{2}\delta_k^j \\
\mathrm{d}y^j(\partial_{z^k}) &= \frac{1}{2i}(\mathrm{d}z^j - \mathrm{d}\bar{z}^j)(\partial_{z^k}) = \frac{1}{2i}\delta_k^j \\
\mathrm{d}x^j(\partial_{\bar{z}^k}) &= \frac{1}{2}(\mathrm{d}z^j + \mathrm{d}\bar{z}^j)(\partial_{\bar{z}^k}) = \frac{1}{2}\delta_k^j \\
\mathrm{d}y^j(\partial_{\bar{z}^k}) &= \frac{1}{2i}(\mathrm{d}z^j - \mathrm{d}\bar{z}^j)(\partial_{\bar{z}^k}) = -\frac{1}{2i}\delta_k^j
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{\partial}{\partial z^j} &= \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right) \\
\frac{\partial}{\partial \bar{z}^j} &= \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right)
\end{aligned}
$$

For a complex-valued function f, in this new basis, we have

$$
\mathrm{d}f = \frac{\partial f}{\partial z^j}\mathrm{d}z^j + \frac{\partial f}{\partial \bar{z}^j}\mathrm{d}\bar{z}^j \tag{3}
$$

and f is holomorphic iff $\frac{\partial f}{\partial \bar{z}^j} = 0$. (If you write this condition out in (x, y) coordinates, you can see the Cauchy-Riemann equations pop out!)

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2.3. The operators ∂, $\overline{\partial}$, and finally Δ.

We come back to the situation of a surface. In the sequence

$$
\Omega_X^0(\mathbf{C}) \xrightarrow{\mathrm{d}} \Omega_X^{1,0} \oplus \Omega_X^{0,1} \xrightarrow{\mathrm{d}} \Omega_X^2(\mathbf{C})
$$

both exterior derivatives split along with the 1-forms.

Definition 2.6. The ∂ and $\overline{\partial}$ operators are the components of d according to the splitting of Ω_X^1. So

$$
\mathrm{d} = \partial + \overline{\partial}
$$

$$
\begin{array}{ccc}
\Omega_X^{1,0} & \xrightarrow{\overline{\partial}} & \Omega_X^2 \\
{\scriptstyle\partial}\uparrow & & {\scriptstyle\partial}\uparrow \\
\Omega_X^0 & \xrightarrow{\overline{\partial}} & \Omega_X^{0,1}
\end{array}
$$

More precisely, acting on 1-forms,

$$
\partial = \pi^{1,0} \circ \mathrm{d} \qquad \overline{\partial} = \pi^{0,1} \circ \mathrm{d}
$$

Definition 2.7. A $(1,0)$-form α is called holomorphic if $\overline{\partial}\alpha = 0$. Such an α has the form $\alpha(z) = f(z)\mathrm{d}z$, where f is a holomorphic function.

So in complex analysis we were integrating $(1,0)$-forms along paths all this time. And (the first) Cauchy's theorem was just a result of Stokes' theorem! For a closed loop γ and a holomorphic function f, we have

$$\int_{\gamma=\partial\Omega} f(z)\ \mathrm{d}z = \int_{\Omega} \mathrm{d}(f\mathrm{d}z) = \int_{\Omega} (\overline{\partial} f)\mathrm{d}\bar{z}\wedge \mathrm{d}z = 0.$$

In the special basis above, $\partial = \mathrm{d}z \wedge \frac{\partial}{\partial z}$ and $\overline{\partial} = \mathrm{d}\bar{z}\wedge\frac{\partial}{\partial \bar{z}}$. The fact that $\mathrm{d}^2 = 0$ translates to

$$\partial^2 = \overline{\partial}^2 = 0$$
$$\partial\overline{\partial} = -\overline{\partial}\partial$$

We look at this last operator in local coordinates:

$$\begin{aligned}\overline{\partial}\partial f &= -\frac{\partial}{\partial\bar{z}}\frac{\partial}{\partial z} f\ \mathrm{d}z\wedge\mathrm{d}\bar{z} = -\frac{1}{4}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f\ (\mathrm{d}x + i\mathrm{d}y)\wedge(\mathrm{d}x - i\mathrm{d}y)\\ &= \frac{i}{2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)\mathrm{d}x\wedge\mathrm{d}y\end{aligned}$$

Therefore, remembering our desired sign convention,

Definition 2.8. Define the <u>laplacian</u> $\Delta : \Omega^0_X \to \Omega^2_X$ by $\Delta = 2i\overline{\partial}\partial$. Notice Δ maps real forms to real forms.

2.4. **Dirichlet energy.**

Continuing with the theme of definitions, we will need one more thing before we solve the equation.

Notice that in defining the laplacian, we used $\mathrm{d}x\wedge\mathrm{d}y$ as our "reference area form". We also learned that

$$\mathrm{d}z\wedge\mathrm{d}\bar{z} = -i\mathrm{d}x\wedge\mathrm{d}y.$$

Definition 2.9. If $\alpha = f(z)\mathrm{d}z$ and $\beta = g(z)\mathrm{d}z$ are $(1,0)$-forms, then define the <u>L^2 inner product</u> by

$$\langle\alpha,\beta\rangle_{L^2} := \int_X i\alpha\wedge\overline{\beta} = \int_X i(f,g)\mathrm{d}z\wedge\mathrm{d}\bar{z} = \int_X (f,g)\mathrm{d}x\wedge\mathrm{d}y$$

where $(f,g) = f\overline{g}$ is the standard hermitian inner product on $\mathbf{C}$. Notice the <u>L^2 norm</u> is then

$$\|\alpha\|^2_{L^2} = \int_X |f|^2\ \mathrm{d}x\wedge\mathrm{d}y.$$

The good thing is that the expression $i\alpha\wedge\overline{\beta}$ is coordinate independent.

This works as an inner product on $(1,0)$-forms. For the laplacian, we will need an energy functional on functions. Note that if f is a real-valued function on X, then

$$\overline{\left(\frac{\partial f}{\partial z}\right)} = \frac{\partial\bar{f}}{\partial\bar{z}} = \frac{\partial f}{\partial\bar{z}}$$

or in the 1-form language,

$$\overline{(\partial f)} = \overline{\partial}\bar{f} = \overline{\partial} f$$

So $\mathrm{d}f$ for a real function written in the $(\mathrm{d}z,\mathrm{d}\bar{z})$ basis is of the form

$$\mathrm{d}f = A\mathrm{d}z + \bar{A}\mathrm{d}\bar{z}, \qquad A = \frac{\partial f}{\partial z}$$

Conversely, for any complex function A, the expression $A\mathrm{d}z + \bar{A}\mathrm{d}\bar{z}$ defines a real 1-form (in the $(\mathrm{d}x, \mathrm{d}y)$ basis):

$$A\mathrm{d}z + \bar{A}\mathrm{d}\bar{z} = A\mathrm{d}(x+iy) + \bar{A}\mathrm{d}(x-iy) = (A+\bar{A})\mathrm{d}x + (A-\bar{A})i\mathrm{d}y = 2\mathrm{Re}(A)\mathrm{d}x - 2\mathrm{Im}(A)\mathrm{d}y$$

i.e. we can identify real 1-forms with $(1,0)$-forms. With this in mind:

Definition 2.10. Define the Dirichlet inner product for $f, g \in \mathbf{C}^\infty(X, \mathbf{R})$ by

$$\langle f, g\rangle_D = \langle \mathrm{d}f, \mathrm{d}g\rangle_{L^2} := 2\langle \partial f, \partial g\rangle_{L^2} = 2i\int_X \partial f \wedge \overline{\partial} g$$

The Dirichlet energy for f is then the associated norm.

$$\|f\|^2 = \langle f, f\rangle_D = 2\int_X \left|\frac{\partial f}{\partial z}\right|^2 \mathrm{d}x \wedge \mathrm{d}y \tag{4}$$

(Why the factor of 2: I think it is related to $a + \bar{a} = 2\mathrm{Re}(a)$).
An important property of the Dirichlet energy is that

$$\begin{aligned}
\langle f, g\rangle_D &= 2i\int_X \partial f \wedge \overline{\partial} g \\
&= 2i\int_X \mathrm{d}f \wedge \overline{\partial} g - \underbrace{\overline{\partial} f \wedge \overline{\partial} g}_{=0} \\
&= 2i\int_X \mathrm{d}(f\overline{\partial} g) - 2i\int_X f\mathrm{d}\overline{\partial} g \\
&= -2i\int_X f\partial\overline{\partial} g \\
&= \int_X f\Delta g.
\end{aligned}$$

3. Back to reality: solving the equation (PDE)

All the functions below are real; we write $C^\infty(X)$ for $C^\infty(X, \mathbf{R})$.

There are three claims in Theorem 1.1.

1. If ρ is a 2-form with $\int_X \rho = 0$, then there exists a function f on X with $\Delta f = \rho$.
2. If f is any function on X, then $\int_X \Delta f = 0$.
3. A solution f to $\Delta f = \rho$ is unique up to adding a constant.

The second point follows from Stokes' theorem,

$$\int_X \partial\overline{\partial} f = \int_X \mathrm{d}\overline{\partial} f = 0.$$

For the third, if $\Delta f = \Delta f' = \rho$, then $g = f - f'$ satisfies $\Delta g = 0$. Then

$$\|\mathrm{d}g\|_{L^2}^2 = \|g\|_D^2 = \int_X g\Delta g = 0$$

so $\mathrm{d}g \equiv 0$. (By (4), and since g is real, $|\partial_z g| \equiv 0$ says that g is constant).

So the main thing to establish is the existence of a solution, given a 2-form ρ with integral 0. We will look to solve the equation "weakly", using the Riesz representation theorem.

By (4), the Dirichlet energy is a seminorm on $C^\infty(X)$; constant functions have norm 0. So we consider the space $\hat{H} := C^\infty(X)/\mathbf{R}$, identifying functions which differ by a constant. With the Dirichlet inner product, $\hat{H}$ becomes a pre-Hilbert space. Let H be the Hilbert space formed by Cauchy completion, so an element of H is represented by a Cauchy sequence in $\hat{H}$.

Notice that $\Delta f = \rho$ if and only if

$$\int_X (\Delta f - \rho)\psi = 0$$

for every test function $\psi \in C^\infty(X)$. Integrating by parts we have

$$\langle f, \psi \rangle_D = \rho(\psi) \tag{5}$$

where ρ denotes the linear functional on $C^\infty(X)$

$$\rho(\psi) := \int_X \rho\psi.$$

(A solution to (5) is called a weak solution of Poisson's equation.) Since $\int_X \rho = 0$, the functional ρ descends to a functional on $\hat{H}$. Furthermore if ρ was bounded, that is if there is a C such that

$$\rho(\psi) \leq C\|\psi\|_D \tag{6}$$

for every $\psi \in \hat{H}$, then ρ would extend to a functional on H, and the following functional analysis tool would apply, solving (5).

Lemma 3.1 (Riesz Representation)**.** *If ρ is a bounded linear functional on a Hilbert space H, then there exists a unique $f \in H$ such that ρ is represented by an inner product with f:*

$$\rho(\phi) = \langle f, \phi \rangle \tag{7}$$

for every $\phi \in H$. Furthermore, $\|f\|_D = \|\rho\|$.

Proof. We will first show the norms must be equal if (7) is true. Recall that $\|\rho\|$ is the operator norm, the smallest constant such that (6) holds. By the representation (7) and Cauchy-Schwarz,

$$\rho(\phi) \leq \|f\|\|\phi\|.$$

Moreover we have equality when $\phi = f$, which proves that $\|f\|$ is the smallest possible constant.

We assume ρ is nonzero. Let $K = \ker\rho$, and S be the unit sphere in H. Since ρ maps to a 1-dimensional space, $K^\perp$ is 1-dimensional. Take any generator g for this space and normalize so that $\|g\| = 1$. Now for any $\phi = k + \lambda g \in H = K \oplus K^\perp$,

$$\rho(\phi) = \rho(k + \lambda g) = \lambda\rho(g) = \langle g, \phi \rangle \rho(g) = \langle f, \phi \rangle$$

for $f := \rho(g)g$. Notice we need the completeness of the Hilbert space to use $K^\perp$. □

Thus we need to show that ρ is indeed a bounded functional. However, this only gives us an element $f \in H$, which is not necessarily even a function. It is represented by a Dirichlet-norm-Cauchy sequence of additive-constant-equivalence classes of functions on X. The final steps are to show that this object is indeed a function on X, and that it is smooth.

3.1. **Bounded functional.**

Let Ω be a bounded convex open set in $\mathbf{R}^2$, with area A and diameter d.

Theorem 3.2. *Let ψ be a smooth function on $\overline{\Omega}$, with average $\overline{\psi} = \frac{1}{A}\int_\Omega \psi$. Then*

$$\left|\psi(x) - \overline{\psi}\right| \leq \frac{d^2}{2A}\int_\Omega \frac{1}{|x-y|}|\nabla\psi(y)| \, \mathrm{d}y.$$

Proof. □

Corollary 3.3.

$$\int_\Omega \left|\psi(x) - \overline{\psi}\right|^2 \, \mathrm{d}x \leq \frac{d^2\pi}{2A}\int_\Omega |\nabla\psi(y)|^2 \, dy$$

Definition 3.4. The convolution of functions f, g is

$$(f * g)(x) = \int_{\mathbf{R}^2} f(y)g(x-y) \, \mathrm{d}y.$$

Lemma 3.5. *If $\|\cdot\|_T$ is any translation invariant norm on functions on $\mathbf{R}^2$, then*

$$\|f * g\|_T \leq |f|_{L^1}\|g\|_T$$

Theorem 3.6. *The functional ρ is bounded.*

3.2. **Regularity: Weyl's Lemma.**

Lemma 3.7. *The weak solution can be identified, up to a constant, with a function in $L^2_{\mathrm{loc}}(X)$*

Theorem 3.8 (Weyl's Lemma)**.** *Let Ω be a bounded open set in $\mathbf{C}$ and ρ be a smooth 2-form on Ω. Suppose f is an L^2 function on Ω such that, for any test function (smooth, compactly supported) on Ω,*

$$\int_\Omega f\Delta\psi = \int_\Omega \psi\rho.$$

Then f is smooth and satisfies $\Delta f = \rho$.

Definition 3.9. Let K be the Newton potential

$$K(x) = \frac{1}{2\pi}\log|x|.$$

Lemma 3.10. *For any smooth function f with compact support in $\mathbf{C}$, the convolution $K * f$ is smooth.*

Lemma 3.11.

(a) *If σ has compact support then $K * (\Delta\sigma) = \sigma$.*
(b) *If f has compact support then $\Delta(K * f) = f$.*

[To be continued.]